The correct way to look at this is to consider it as a singular perturbation problem, whose outer solution is given by the point-mass results.

It has long been known that if an artificial launcher length, longer than the real physical launcher length is used the singularity in the point-mass equations can be avoided and reasonable looking results obtained. The purpose of this Note is to solve the singular perturbation problem and to develop an artificial launcher length, for use in the point-mass model, which will provide valid solutions everywhere except very near the launcher.

Solution

The point of departure is the solution given[1] for the gravity turn. Since gravity acts as an Earth fixed disturbance, the solution to the inner problem can be found on a planar, nonrolling basis. The simplest dynamical model is based on the body axis equations

$$\dot{w} - \dot{\theta} u = g \sin\gamma \tag{1}$$

$$I\ddot{\theta} = M_\alpha \alpha \tag{2}$$

$$M_\alpha = -I\lambda u^2 \tag{3}$$

$$\alpha \cong \frac{w}{u} \tag{4}$$

$$\gamma = \theta - \alpha \tag{5}$$

where g = acceleration due to gravity; $u, w = x$ and z velocity components; γ = flight path angle, I = pitch moment of inertia; λ = initial pitch wave number, and θ = inertial pitch angle. Assuming constant axial acceleration, and replacing $\sin\gamma$ by its initial value γ_0 in Eq. (1), the resulting flight path angle history near launch is given[1] as

$$\frac{\gamma_3}{\gamma_0}(s) = 1 + \frac{g}{a}\left[\frac{\pi}{2\lambda s}\right]^{1/2}\Big[\sin\lambda s\,\left(S(\lambda s) - S(\lambda L)\right) + \cos\lambda s\left(C(\lambda s) - C(\lambda L)\right)\Big] + \pi\frac{g}{a}\Big[J(\lambda s) - J(\lambda L) + C(\lambda s)S(\lambda L) - C(\lambda L)S(\lambda s)\Big] \tag{6}$$

where L = launcher length, s = distance along flight path,

$$C(x) + iS(x) = \int_0^x \frac{e^{it}}{[2\pi t]^{1/2}}\,dt = \text{Fresnel integrals, and}$$

$$J(x) = \int_0^x \frac{\sin t\, C(t) - \cos t\, S(t)}{[2\pi t]^{1/2}}\,dt$$

Note that γ_3 is the flight path angle solution obtained from the above three degree of freedom (DOF) analysis. The corresponding two degrees of freedom, or zero α point mass, solution is given by

$$\frac{\gamma_2}{\gamma_0}(s) = 1 + \frac{g}{2a}\log\frac{\lambda s}{\lambda L} \tag{7}$$

for small s.

The outer limit of the inner solution is found using results obtained from Ref. 2. When the inner limit of the outer solution, Eq. (7), is matched to the outer limit of the inner solution, both show the logarithmic form previously indicated, demonstrating procedural validity.

The matching of the two solutions leads to the result for the launcher length L_2 to be used in 2 DOF, or point mass, simulations which will have the same asymptotic behavior as 3 DOF solutions for launcher length L_3. It is found to be

Table 1 Incremental launcher length

λL_3	$\lambda(L_2 - L_3)$
0.000	0.1403
0.001	0.1509
0.002	0.1550
0.005	0.1624
0.010	0.1702
0.020	0.1796
0.050	0.1939
0.100	0.2038
0.200	0.2085
0.500	0.1967
1.000	0.1678
2.000	0.1245

$$\frac{L_2}{L_3} = \frac{1}{4\lambda L_3}\exp\left\{-\Gamma + 2\pi J(\lambda L_3) + \pi\left[C(\lambda L_3) - S(\lambda L_3)\right]\right\} \tag{8}$$

where $\Gamma = 0.5882157...$ which is Euler's or Mascheroni's constant.

Results

Equation (8) may be evaluated numerically using the integral tables of Ref. 1. When this is done, it is found that the dimensionless increment in launcher length, $\lambda(L_2 - L_3)$, is the most nearly constant of the output variables. The numerical data are displayed in Table 1. These calculations place the customary practice of using a larger than life size launcher in point-mass trajectory simulations on a firm theoretical foundation.

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Imperfection Sensitivity and Isoperimetric Variational Problems in Stability Theory

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Introduction

THE investigation of shell buckling has occupied an increasingly prominent place in aeronautical and

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hydronautical studies for more than twenty years. The first of several reasons for this is the increase in fields of application; e.g.: water tanks, ballistic missile bulkheads, re-entry and entry heat shields, space and undersea vehicles, and landing capsules for planets. In addition, the simple nonlinear equation of cylindrical and spherical shell structures was used as a test for the potentialities of the electronic computer. However, in our view the most important reason is the notoriously strong discrepancy between theoretical and experimental results of shell buckling loads.[1] These loads were often found to be less than half the theoretical prediction. The buckling pattern was essentially different from that expected theoretically and the results show strong scatter. Also, buckling occurred explosively, often leading to the destruction of the specimen.

This problem resisted all attempts at explanations until it became one of the most challenging problems in the elasticity theory. Von Karman et al.[2] was the first to show that an explanation is only possible within a nonlinear theory and that determining the buckling load alone is not sufficient for shell structures. This is because another equilibrium state exists at a finite distance from the buckling load which involves a lower energy level. The shell was then supposed to jump into a corresponding configuration without reaching the buckling load predicted by the linear theory. The nonlinear terms causing this unstable post buckling path, which has a minimum at approximately the same critical value observed experimentally, was introduced to the analysis through the stretching energy of the shell middle surface. This last point is very important. Von Karman's large-deflection approach dominated investigations on shell buckling for many years although it is not completely satisfactory.[3,4] However, after 50 years of considerable research effort, initial imperfection[1,5-7] is recognized as the major cause for the discrepancy between theory and experiment. The initial post buckling approach, pioneered by Koiter[5] in Holland shortly after von Karman's contribution, remained unknown for a long time and was not recognized by other research workers until 1963.[6,7] Nevertheless, the incorporation of imperfection sensitivity into engineering practice has not been accomplished because of many difficulties associated with determining the actual imperfection.[3]

In the meantime, a simple concept of estimating the buckling load was developed.[3,4,8-13] This work is mainly based upon the work in the early sixties of the mathematician, Pogorelov,[8,9] who developed an ingenious solution for shell buckling based on geometrical arguments. This method was further developed by giving a mechanical interpretation to the geometrical arguments.[3,4,10-12] There it was argued that due to the energy balance, the buckling mode will be isometric and will be triggered in small regions of the shell. This interpretation also introduced the notion of attenuated local isometric buckling. The results obtained were in excellent agreement with experimental evidence for several particular cases. Thus, the discrepancy between theoretical and experimental buckling loads was explained by the severe unstable post buckling behavior. This, in turn, was attributed more or less to the role of the stretching energy of the shell middle surface.

In developing simple engineering approaches to essentially complicated problems, one must beware of oversimplification and must know the limits of this simplified theory. It is difficult to see why there should be no other reason for the unstable post buckling behavior besides the distabilizing effect of the stretching energy. Some authors even went so far as to consider that neglecting the stretching energy contribution to the buckling load (in the case of axially compressed cylinders this is 50%) would lead to a lower bound solution. However, a structure of piecewise stable symmetrical post buckling could behave completely differently under nonlinear auxiliary constraint or compatibility connecting the different parts of the structure. This explains some work carried out long ago by Chwalla[14] on the stability of frames. This structure is essentially axially inextensional and seems to display a shell-like behavior insofar as it is sensitive to imperfection. The present short analysis thus serves as a reminder of the limitation of the simplified engineering approaches to shell buckling as well as a warning of oversimplification. It shows that an inextensional structure can also have unstable post buckling behavior due to nonlinear terms introduced through auxiliary conditions. It also shows that an asymmetric point of bifurcation can be displayed by frames, and that the common belief that frames always buckle under neutral equilibrium conditions is erroneous for extensional or inextensional middle axes.

Analysis

Consider the frame shown in Fig. 1. When the axial load P reaches a critical value, the column of the frame will buckle in a sinusoidal form which is adequately approximated by

$$\omega_1 = a_1 \sin(\pi/l)x_1 \tag{1}$$

and

$$\omega_2 = a_2 \sin(\pi/l)x_2 \tag{2}$$

where x_1, x_2 are coordinates in the longitudinal and opposite directions, and w_1, is the vertical and w_2 is the horizontal deflected shapes, respectively. Writing the energy functional as the sum of the bending and loading energy, we obtain

$$V = \frac{EI}{2}\int_0^{\ell}(x_1^2 + x_2^2)\,\mathrm{d}x_{1,2} - P\Delta\ell_1 \tag{3}$$

where $\Delta\ell_1$ is the vertical displacement of the corner of the frame, EI is the bending stiffness, and x_i is the change of curvature. Expressing x_i and $\Delta\ell_i$ in terms of the displacement component, we have

$$x_i = \sin^{-1} w_i^* \tag{4}$$

$$\Delta\ell_i = \ell_i - \int_0^{\ell_i}(1 - \overset{*}{w}{}_i^2)^{1/2}\,\mathrm{d}x_i \tag{5}$$

where

$$(^*) = \mathrm{d}(\ )/\mathrm{d}x_i$$

Inserting this in the energy functional and retaining up to fourth-order terms only, we have

$$V = \frac{EI}{2}\int_0^{\ell}(w_1''^2 + w_1''^2 w_1'^2)\,\mathrm{d}x_1 + \int_0^{\ell}\left(-\frac{1}{2}Pw_1'^2 - \frac{P}{8}w_1'^4\right)\mathrm{d}x_1 + \frac{EI}{2}\int_0^{\ell}(\ddot{w}_2^2 + \ddot{w}_2^2\dot{w}_2^2)\,\mathrm{d}x_2 \tag{6}$$

where $(\,')=\mathrm{d}(\,)/\mathrm{d}x_i$ and $(\dot{\ })=\mathrm{d}(\,)/\mathrm{d}x_2$. The relationship between w_1 and w_2 is given by the compatability condition at the rigid corner which requires that the sum of the change in the slope at the corner vanishes

$$\psi_1 - \psi_2 = 0 \tag{7}$$

With

$$\psi_1 \cong \varphi_1 + \Delta\ell_2/\ell_1 \tag{8}$$

and

$$\psi_2 \cong \varphi_2 + \Delta\ell_1/\ell_2 \tag{9}$$

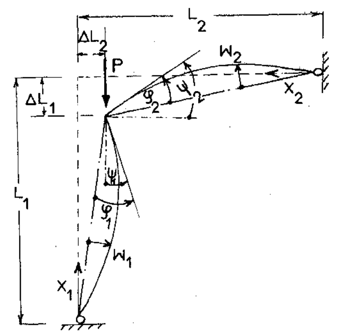

Fig. 1 Inextensional frame under axial load.

where

$$\varphi_I = w_I'|_{(x=\ell)} \quad (10)$$

and

$$\varphi_2 = \dot{w}_2|_{(x=\ell)} \quad (11)$$

are the angles of rotation with $x_I = x_2$. Taking up to second-order terms, we obtain the auxiliary condition

$$a_2 = a_I - a_I^2(\pi/2\ell) \quad (12)$$

which, together with Eq. (6), forms an isoperimetric variational problem. Inserting Eqs. (1), (2), and (12) into Eq. (6), the isoperimetric problem can be reduced to a free variational problem given by

$$V = \frac{EI}{2}\left(a_I^2\frac{\pi^4}{\ell^3} - \frac{P}{EI}a_I^2\frac{\pi^2}{2\ell} - a_I^3\frac{\pi^5}{2\ell^4}\right) \quad (13)$$

where the fourth-order terms have been neglected compared with the third-order terms. Differentiating Eq. (13) with respect to a_I and changing the perturbation parameter from a_I to φ_I (using $\varphi_I = \pi/\ell a_I$), the equation governing the initial post buckling behavior is obtained as

$$P = 19.7(EI/\ell^2) - \tfrac{3}{4}(\pi/\ell)^2 EI\varphi_I \quad (14)$$

or‡

$$P/P^c = 1 - 0.375\varphi_I \quad (15)$$

Thus, the post buckling behavior is unstable due to a non-vanishing initial post buckling slope. In other words, the system possesses an asymmetrical point of bifurcation and is, therefore, imperfection sensitive. The behavior resembles, in principle, the behavior of the cylindrical shell under axial pressure and the spherical shell under external pressure where the initial slope of the post buckling path was found to be nonzero (Fig. 2). It is important to note that the potential energy of the frame involves no stretching energy at all and that the nonvanishing third-order terms which caused the non-vanishing initial post buckling slope were introduced into the energy functional through the auxiliary nonlinear compatibility condition. This is in clear contrast to the elastic problems,[7] where the rise or fall of the post buckling path is due to the taking of the higher-order terms of the rotation into account.

Conclusion

The preceding analysis shows two important points. First, unstable post buckling behavior can arise due to an isoperimetric condition. This means that many of the recent publications concerning shell buckling require some modification, as they attribute unstable behavior solely to the role of the stretching energy.[10,15] Second, the buckling of

‡Equation (15) is almost identical with that obtained by Koiter ($P/P^c = 1 - 0.3805\varphi_1$) using an exact and, therefore, more complicated analysis.

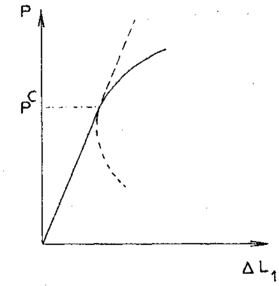

Fig. 2 Post critical behavior of the inextensional frame.

equally extensional or inextensional frames does not, in general, occur under neutral equilibrium conditions.

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Method of Integral Relations and Triple-Point Location in Impinging Jets

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Nomenclature

M_N = nozzle exit Mach number

p_a = ambient pressure

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